On a mean-value theorem of Schwarz—Stieltjes.*)

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Let $f(x)$ be a single-valued function of the real variable x, which has derivatives of the first n orders throughout the open interval (a, b). Let the real numbers $a_0, \ldots, a_k$ and the positive integers $m_0, \ldots, m_k$ be chosen such that

$$a < a_0 < a_1 < \cdots < a_k < b$$

and

(1) $$m_0 + m_1 + \cdots + m_k = n + 1.$$

If, following N. E. NÖRLUND[1]), we denote by

$$[\overbrace{a_0, \ldots, a_0}^{m_0 \text{ times}}, \overbrace{a_1, \ldots, a_1}^{m_1 \text{ times}}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k \text{ times}}; f(x)]$$

the coefficient of x^n in the polynomial $H(x)$ of order $\leqq n$ satisfying the interpolatory conditions

(2) $$H(a_i) = f(a_i),\ H'(a_i) = f'(a_i), \ldots, H^{(m_i-1)}(a_i) = f^{(m_i-1)}(a_i)$$
$$(i = 0, 1, \ldots, k)$$

of CH. HERMITE[2]), then the mean-value theorem

(3) $$[\overbrace{a_0, \ldots, a_0}^{m_0 \text{ times}}, \overbrace{a_1, \ldots, a_1}^{m_1 \text{ times}}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k \text{ times}}; f(x)] = \frac{f^{(n)}(\xi)}{n!}$$

of SCHWARZ—STIELTJES[3]) is valid, where $a_0 < \xi < a_k$.

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*) Abbreviated version of a previous paper of the author in Hungarian: A differenciálszámítás középértéktételével kapcsolatos kérdésekről, *Mathematikai és Physikai Lapok*, **33** (1926), 150—180.

1) N. E. NÖRLUND, *Leçons sur les séries d'interpolation* (Paris, 1926), p. 7—8.

2) CH. HERMITE, Sur la formule d'interpolation de Lagrange, *Journal für die reine und angewandte Math.*, **84** (1878), 70—79, in particular p. 70, or *Oeuvres de Charles Hermite*, III (Paris, 1912), p. 432—443, in particular p. 432.

3) H. A. SCHWARZ, Démonstration élémentaire d'une propriété fondamentale des fonctions interpolaires, *Gesammelte mathematische Abhandlungen II* (Berlin, 1890), p. 307—308, in particular p. 308; further T. J. STIELTJES, A propos de la formule d'interpolation de Lagrange, *Oeuvres complètes de Thomas Jan Stieltjes* I (Groningen, 1914), p. 47—60, in particular p. 56. H. A. SCHWARZ stated the theorem only in the case $k = n$ and supposed the continuity of $f^{(n)}(x)$.

In the case

$$k=n,\ a_0=\alpha,\ a_1=\alpha+h,\ldots,a_n=\alpha+nh$$

(3) can be written in the form

$$(M_n)\qquad \Delta^n f(\alpha)=h^n f(\alpha+\tau n h)\qquad (0<\tau<1)$$

while for

$$k=1,\ a_0=\alpha,\ a_1=\alpha+h,\ m_0=n,\ m_1=1$$

from (3) it follows TAYLOR's mean-value theorem with LAGRANGE's form of the remainder:

$$(T_n)\quad f(\alpha+h)=f(\alpha)+\frac{h}{1!}f'(\alpha)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(\alpha)+\frac{h^n}{n!}f^{(n)}(\alpha+\vartheta h)$$
$$(0<\vartheta<1)$$

It has been proved by R. ROTHE[4]), that in (M_n)

$$(4)\qquad \tau\to\frac{1}{2}\quad\text{for}\quad h\to 0$$

and in (T_n)

$$(5)\qquad \vartheta\to\frac{1}{n+1}\quad\text{for}\quad h\to 0,$$

assumed in both cases that $f^{(n+1)}(x)$ exists in the neighbourhood of the point α, and is continuous and $\neq 0$ in α.[5]) I have shown[6]), that it is sufficient to suppose the existence of $f^{(n+1)}(\alpha)\neq 0$.

In the present paper I shall prove first the following theorem, which contains the above theorems (4) and (5):

Theorem I. *Suppose the points a_0, a_k tend to the point α of the open interval (a, b) with the restriction*

$$(6)\qquad \left|\frac{a_0-\alpha}{a_k-a_0}\right|\leqq L,\qquad \left|\frac{a_k-\alpha}{a_k-a_0}\right|\leqq L.$$

Then for the mean-value ξ in (3) *we have*

$$(7)\qquad \frac{1}{a_k-a_0}\left(\xi-\frac{m_0a_0+m_1a_1+\cdots+m_ka_k}{m_0+m_1+\cdots+m_k}\right)\to 0,$$

assumed that $f^{(n+1)}\alpha$ exists and is $\neq 0$.

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[4]) R. ROTHE, Zum Mittelwertsatze der Differentialrechnung, *Math. Zeitschrift*, 9 (1921), 300—325, in particular p. 314, further p. 309—310.

[5]) The latter theorem has been proved by M. BEKE, *Differenciál- és integrálszámitás* I. (in Hungarian), (Budapest, 1910), p. 197.

[6]) PAUL SZÁSZ, Über einen Mittelwertsatz, *Math. Zeitschrift*, 25 (1926), 116—120. Later the same was proved for TAYLOR's mean-value theorem (T_n) by CHR. Y. PAUC, see OTTO HAUPT—GEORG AUMANN, *Differential- und Integralrechnung*, 2. Aufl. unter Mitwirkung von CHRISTIAN Y. PAUC, II. Band (Berlin, 1950), p. 42.

I need the following

Lemma. *We have*

(8) $$[\overbrace{a_0, \ldots, a_0}^{m_0 \text{ times}}, \overbrace{a_1, \ldots, a_1}^{m_1 \text{ times}}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k \text{ times}}; x^{n+1}] = m_0 a_0 + m_1 a_1 + \cdots + m_k a_k.$$

Proof of the lemma. In (2) choose $f(x) = x^{n+1}$. Then the roots of the algebraic equation

(9) $$x^{n+1} - H(x) = 0$$

are

$$\overbrace{a_0, \ldots, a_0}^{m_0 \text{ times}}, \overbrace{a_1, \ldots, a_1}^{m_1 \text{ times}}, \ldots, \overbrace{a_k, \ldots, a_k}^{m_k \text{ times}}$$

every root being taken according to its multiplicity. The sum of these roots of the equation (9) is equal to the coefficient of x^n in $H(x)$, i.e. to the left-hand side of (8), which proves the Lemma.

Proof of theorem I. Let the mean-value theorem (3) be applied to the function

$$f(x) - \frac{f^{(n+1)}(\alpha)}{(n+1)!} x^{n+1}.$$

On the basis of the above Lemma it follows with regard to (3)

$$\frac{f^{(n)}(\xi)}{n!} - \frac{f^{(n+1)}(\alpha)}{(n+1)!}(m_0 a_0 + m_1 a_1 + \cdots + m_k a_k) = \frac{f^{(n)}(\xi')}{n!} - \frac{f^{(n+1)}(\alpha)}{n!}\xi'$$
$$(a_0 < \xi < a_k, \ a_0 < \xi' < a_k),$$

or written in another form

(10) $$f^{(n+1)}(\alpha)\left(\xi - \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{n+1}\right) = \\ = f^{(n)}(\xi') - f^{(n)}(\xi) + f^{(n+1)}(\alpha)(\xi - \xi').$$

But we have by the definition of the derivative

$$f^{(n)}(x) = f^{(n)}(\alpha) + \{f^{(n+1)}(\alpha) + \varepsilon(x)\}(x - \alpha)$$

with

(11) $$\lim_{x=\alpha} \varepsilon(x) = 0.$$

Therefore we obtain from (10)

$$f^{(n+1)}(\alpha)\left(\xi - \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{n+1}\right) = (\xi' - \alpha)\varepsilon(\xi') - (\xi - \alpha)\varepsilon(\xi),$$

hence with respect to (1)

$$\frac{1}{a_k - a_0}\left(\xi - \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{m_0 + m_1 + \cdots + m_k}\right) = \tag{12}$$

$$= \frac{1}{f^{(n+1)}(\alpha)}\left\{\frac{\xi' - \alpha}{a_k - a_0}\varepsilon(\xi') - \frac{\xi - \alpha}{a_k - a_0}\varepsilon(\xi)\right\}.$$

In consequence of the restriction (6) we have however

$$\left|\frac{\xi - \alpha}{a_k - a_0}\right| < L, \quad \left|\frac{\xi' - \alpha}{a_k - a_0}\right| < L.$$

Thus on the basis of (11), from (12) it follows (7), q. e. d.

Next we make another remark concerning the mean-value ξ in the formula (3). In case of a polynomial $f(x)$ of the exact order $n+1$, it follows at once that

$$\xi = \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{m_0 + m_1 + \cdots + m_k}.$$

This fact can be generalized in the following manner:

Theorem II. *Let $f^{(n)}(x)$ monotone throughout the interval $a_0 < x < a_k$, further let the difference-quotient*

$$\varphi(u, v) = \frac{f^{(n)}(u) - f^{(n)}(v)}{u - v}$$

be bounded for $a_0 < u < v < a_k$, and M, μ denote the upper and lower bound of $|\varphi(u, v)|$, respectively. Then in case $M \neq 0$ we have

$$\frac{1}{a_k - a_0}\left|\xi - \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{m_0 + m_1 + \cdots + m_k}\right| \leqq 1 - \frac{\mu}{M}. \tag{13}$$

Equality is valid if and only if $M = \mu$.

Proof. Similarly to the proof of (10) we obtain

$$M\left(\xi - \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{n+1}\right) = f^{(n)}(\xi'') - f^{(n)}(\xi) + M(\xi - \xi'') \tag{14}$$

$$(a_0 < \xi < a_k, \ a_0 < \xi'' < a_k).$$

In case $\xi'' = \xi$ the assertion is obvious. Now let $\xi'' \neq \xi$. From (14) and (1) it follows in consequence of

$$\left|\frac{\xi - \xi''}{a_k - a_0}\right| < 1$$

that

$$\frac{1}{a_k - a_0}\left|\xi - \frac{m_0 a_0 + m_1 a_1 + \cdots + m_k a_k}{m_0 + m_1 + \cdots + m_k}\right| \leqq \frac{1}{M}\left(M - \frac{f^{(n)}(\xi) - f^{(n)}(\xi'')}{\xi - \xi''}\right). \tag{15}$$

It can be assumed that $f^{(n)}(x)$ does not decrease, ξ, M, μ being the same for $f(x)$ as for $-f(x)$. Then we have

$$\mu \leqq \frac{f^{(n)}(\xi)-f^{(n)}(\xi'')}{\xi-\xi''} \leqq M \tag{16}$$

and from (15), (16) it follows (13). In (15) equality is valid only in the case if the right-hand side vanishes. Consequently, in (13) equality holds if and only if $M=\mu$. Thus the proof of Theorem II is complete.

With the help of Theorem II it is easy to see, for ex., that in the well-known formula

$$\log(N+\nu)=\log N+\nu\{\log(N+1)-\log N\}-\frac{\nu(\nu+1)}{2}\frac{1}{\xi^2}$$
$$(N>0,\ 0<\nu<1,\ N<\xi<N+1)$$

we have

$$\xi=N+\frac{\nu+1}{3}+\sigma$$

with

$$|\sigma|<\frac{3}{N}+\frac{3}{N^2}+\frac{1}{N^3}.$$

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